Engineering Notes

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New Form for the Optimal Rendezvous Equations Near a Keplerian Orbit

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Introduction

IN 1954 Lawden[1] introduced the integral

$$I(\theta) = \int_{\theta_0}^{\theta} \frac{d\zeta}{(1 + e\cos\zeta)^2 \sin^2\zeta} \tag{1}$$

in order to evaluate his *primer vector* during a null thrust (Keplerian) interval, where θ denotes the true anomaly at time t and θ_0 its initial value. In 1963 this integral appears again in his book[2] to determine the primer vector through the solution of his differential Eqs. (5.30–5.32).

These same equations, with only minor modifications, appeared at approximately the same time in the linearization of De Vries[3] to describe the relative motion of two nearby points in elliptical orbits. Except for changes in the coordinate systems, a nonhomogeneous version of these linearized equations was used by Tschauner and Hempel[4-7] to describe the rendezvous of a spacecraft with a target in elliptical orbit. Restricted or generalized forms of these equations were employed by Shulman and Scott,[8] Euler and Shulman,[9] and Euler[10] for rendezvous of a spacecraft with an object in elliptical orbit. The approach of Tschauner and Hempel, as used by Weiss[11] in 1981, was found to be effective in constructing two-impulse solutions to rendezvous problems can occur where the true anomaly is near zero. These problems involving objects in elliptical orbits of high eccentricity.[12] In all of these studies, the solution of Lawden's equations was not investigated through the use of the integral $I(\theta)$ of Eq. (1).

Eckel's paper[13] of 1982 returns to the integral $I(\theta)$ with Lawden's equations to determine the primer vector and solve the problem of optimal impulsive transfer between noncoplanar elliptical orbits. Recently, Carter and Humi[14] and Carter[15] used this integral in solving Lawden's equations to determine both the primer vector and the structure of an optimal rendezvous of a spacecraft with an object near a point in general Keplerian orbit.

Although the integral $I(\theta)$ appears naturally in solving Lawden's differential equation, this integral is singular at values where the true anomaly is a multiple of π. Even though these are removable singularities, they may lead to computational instabilities in the solution. Especially bothersome is the fact that computational problems can occur where the true anomaly is near zero. These problems are avoided in the work of solution that does not involve $I(\theta)$, but their work is confined to elliptical orbits.

The purpose of this Note is to modify the original form by replacing $I(\theta)$ by a related integral $J(\theta)$, thereby removing all singularities and computational instabilities. The resulting solution, in terms of $J(\theta)$, is identical for hyperbolic, parabolic, or noncircular elliptic orbits, but the particular case determines the nature of the closed-form evaluation of $J(\theta)$.

Application of this work to actual problems usually involves the solution of a two-point boundary-value problem and is not considered here. Although we emphasize the case of bounded thrust, the unbounded thrust case can also be investigated through the use of the simpler equations for unpowered flight, which we also present. If the maximum number of impulses is known for an optimal rendezvous in this case, the two-point boundary-value problem is reduced to a problem of parameter optimization on the velocity increments and their locations. We conjecture that the maximum of impulses for this problem is four. If the number of impulses is restricted to two, the problem is solved by methods similar to those of Weiss et al.[11,12] For the case of bounded thrust, a method such as that used for the rendezvous problem near circular orbit[16] can be applied with starting iteratives obtained from the related unbounded thrust case.

Rendezvous Equations

We consider a rotating coordinate frame centered at a point moving in a Keplerian orbit about a central attractive body. The independent variable is the true anomaly θ defined on the closed interval $\theta_0 \le \theta \le \theta_f$, which we denote by $\boldsymbol{\Theta}$. All vectors are assumed to be elements of three-dimensional Euclidean space. The position vector $\boldsymbol{x}(\theta) = [x_1(\theta), x_2(\theta), x_3(\theta)]$ of the spacecraft in this coordinate system is transformed to the vector $\boldsymbol{y}(\theta) = [y_1(\theta), y_2(\theta), y_3(\theta)]$ by the equation

$$\boldsymbol{y}(\theta) = (1 + e\cos\theta)\boldsymbol{x}(\theta) \tag{2}$$

where e denotes the eccentricity of the Keplerian orbit.

This development, which is presented in detail in previous work,[14] results in the following transformed linearized equations of the spacecraft:

$$y_1''(\theta) = 2y_2'(\theta) + a_1(\theta) \tag{3a}$$

$$y_2''(\theta) = \frac{3}{1 + e\cos\theta} y_2(\theta) - 2y_1'(\theta) + a_2(\theta) \tag{3b}$$

$$y_3''(\theta) = -y_3(\theta) + a_3(\theta) \tag{3c}$$

where the prime indicates differentiation with respect to θ, and the vector $\boldsymbol{a}(\theta) = [a_1(\theta), a_2(\theta), a_3(\theta)]$ is given by

$$\boldsymbol{a}(\theta) = \frac{bu(\theta)/m(\theta)}{(1 + e\cos\theta)^3} \tag{4}$$

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In this expression the positive constant b is $L_0^6 T_m/\mu^4$, where L_0 is the magnitude of the constant angular momentum of the object in Keplerian orbit divided by its mass, T_m is the maximum magnitude of the thrust of the spacecraft, and μ is the product of the universal gravitational constant and the mass of the central body of attraction. The vector $\boldsymbol{u}(\theta) = [u_1(\theta), u_2(\theta), u_3(\theta)]$ represents the normalized thrust of the spacecraft and is subject to the constraint

$$|\boldsymbol{u}(\theta)| \leq 1 \tag{5}$$

where the notation $|\ |$ indicates the Euclidean norm or magnitude of a vector. Assuming that the exhaust velocity of the propellent of the spacecraft is a positive constant γ, the mass of the spacecraft $m(\theta)$ satisfies the equation

$$m'(\theta) = \frac{-k\ |\boldsymbol{u}(\theta)|}{(1 + e\cos\theta)^2} \tag{6}$$

where the positive constant k is $\mu^2 b/(L_0^3\gamma)$. Equations (3) are essentially the equations of Tschauner and Hempel,[4] and their homogeneous form represents essentially the equations of De Vries[3] and Lawden.[2]

We define *the class of admissible control functions* as the set of all Lebesgue measurable vector valued functions that satisfy Inequality. (5) a.e. on $\boldsymbol{\Theta}$. The optimal rendezvous problem associated with a point in Keplerian orbit is defined as the determination of an admissible control function $\boldsymbol{u}$ that maximizes the terminal spacecraft mass $m(\theta_f)$ subject to the conditions of Eqs. (3-6), that are valid a.e. on $\boldsymbol{\Theta}$ and the end conditions

$$y(\theta_0) = y_0, \qquad y'(\theta_0) = v_0, \qquad m(\theta_0) = m_0 \tag{7}$$

$$y(\theta_f) = y_f, \qquad y'(\theta_f) = v_f \tag{8}$$

where the vectors y_0 and y_f define the initial and terminal values of the transformed position y, v_0 and v_f define the initial and terminal values of its derivative, and m_0 defines the initial mass of the spacecraft.

Necessary conditions that define the structure of a solution of this optimal rendezvous problem have been determined.[15] For noncircular Keplerian orbits, the principal result that must be satisfied by an optimal admissible control function $\boldsymbol{u}$ is as follows.

Theorem: Either $\boldsymbol{Q}(\theta) = 0$ for each $\theta \in \boldsymbol{\Theta}$ and $\boldsymbol{u}(\theta) = 0$ a.e. on $\boldsymbol{\Theta}$ or else $\boldsymbol{Q}$ is nonzero except at finitely many values and

$$\boldsymbol{u}(\theta) = -(\boldsymbol{Q}(\theta)/|\boldsymbol{Q}(\theta)|)f(\theta) \tag{9}$$

a.e. on $\boldsymbol{\Theta}$, where

$$f(\theta) = \begin{cases} 0, & s(\theta) > 0 \\ 1, & s(\theta) < 0 \end{cases} \tag{10}$$

except at the zeros of s, which are finitely many.

In this statement $\boldsymbol{Q}(\theta) = [Q_1(\theta), Q_2(\theta), Q_3(\theta)]$ is a type of primer vector that has been determined[14] as follows:

$$Q_1(\theta) = -c_1 r(\theta) - c_2\left[r(\theta)I(\theta) + \frac{\cot\theta}{r(\theta)}\right] - c_3\sin\theta\left[1 + \frac{1}{r(\theta)}\right] + \frac{c_4}{r(\theta)} \tag{11a}$$

$$Q_2(\theta) = [c_1 + c_2 I(\theta)]e\sin\theta - c_3\cos\theta \tag{11b}$$

$$Q_3(\theta) = \frac{c_5\sin\theta + c_6\cos\theta}{r(\theta)} \tag{11c}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are arbitrary constants,

$$r(\theta) = 1 + e\cos\theta \tag{12}$$

and $I(\theta)$ is Lawden's integral defined by Eq. (1). It is known[14] that $r(\theta) > 0$ for each $\theta \in \boldsymbol{\Theta}$. The real-valued function s that appears in Eq. (10) is called the "switching function" and has been defined[15] by

$$s(\theta) = l_0 - kl(\theta) - b\,|\boldsymbol{Q}(\theta)|/m(\theta) \tag{13}$$

where l_0 is any nonnegative real number, and $l(\theta)$ satisfies the equation

$$l'(\theta) = -\frac{b\,|\boldsymbol{Q}(\theta)|f(\theta)}{r(\theta)^2 m(\theta)^2} \tag{14}$$

a.e. on $\boldsymbol{\Theta}$.

It is necessary that an optimal admissible control function $\boldsymbol{u}$ satisfy Eqs. (9-14). The mass of the spacecraft is determined from Eq. (6), and Eq. (4) defines the vector function $\boldsymbol{a}$, which is input to the system of Eqs. (3).

The general solution of Eqs. (3) has been found[14] in terms of $\boldsymbol{a}(\theta)$ and the integral $I(\theta)$ and is as follows:

$$y_1(\theta) = -b_1 r(\theta)^2 - b_2[r(\theta)^2 I(\theta) + \cot\theta] - \frac{r(\theta)^2}{e}\int_{\theta_0}^{\theta}\frac{L(\zeta)\mathrm{d}\zeta}{r(\zeta)^2\sin^2\zeta} + \int_{\theta_0}^{\theta}\left[\frac{L(\zeta)}{e\sin^2\zeta} + r_1(\zeta)\right]\mathrm{d}\zeta + b_4 \tag{15a}$$

$$y_2(\theta) = r(\theta)\sin\theta\left[b_1 e + b_2 e I(\theta) + \int_{\theta_0}^{\theta}\frac{L(\zeta)\,\mathrm{d}\zeta}{r(\zeta)^2\sin^2\zeta}\right] \tag{15b}$$

$$y_3(\theta) = [b_5 + L_1(\theta)]\cos\theta + [b_6 + L_2(\theta)]\sin\theta \tag{15c}$$

where

$$L(\theta) = \int_{\theta_0}^{\theta}[a_2(\zeta) - 2r_1(\zeta)]r(\zeta)\sin\zeta\,\mathrm{d}\zeta \tag{16}$$

$$r_1(\theta) = \int_{\theta_0}^{\theta} a_1(\zeta)\,\mathrm{d}\zeta + b_3 e \tag{17}$$

$$L_1(\theta) = -\int_{\theta_0}^{\theta} a_3(\zeta)\sin\zeta\,\mathrm{d}\zeta \tag{18}$$

$$L_2(\theta) = \int_{\theta_0}^{\theta} a_3(\zeta)\cos\zeta\,\mathrm{d}\zeta \tag{19}$$

and b_1, b_2, b_3, b_4, b_5, and b_6 are arbitrary constants. To solve practical problems these constants and their counterparts in Eqs. (11) must be determined through the solution of a two-point boundary-value problem.

It is known from the form of the switching function Eq. (13) and from Eqs. (9) and (10) that an optimal solution must consist of finitely many intervals of full thrust and coast.[15] During an unpowered interval, Eqs. (15) reduce to the following:

$$y_1(\theta) = -b_1 r(\theta)^2 - b_2[r(\theta)^2 I(\theta) + \cot\theta] - b_3\sin\theta[1 + r(\theta)] + b_4 \tag{20a}$$

$$y_2(\theta) = e r(\theta)\sin\theta[b_1 + b_2 I(\theta)] - b_3 r(\theta)\cos\theta \tag{20b}$$

$$y_3(\theta) = b_5\cos\theta + b_6\sin\theta \tag{20c}$$

These equations can be used to define the relative motion of two nearby points in noncircular Keplerian orbits by transforming from $y(\theta)$ to $\boldsymbol{x}(\theta)$ using Eq. (2) and generalizes the work of De Vries.[3] If this is done, the form of $\boldsymbol{x}(\theta)$ is identical to that of $\boldsymbol{Q}(\theta)$ in Eq. (11). This vector $\boldsymbol{x}(\theta)$ is also useful in the investigation of unbounded thrust problems.

A New Form of the Rendezvous Equations

Equations (11), (15), and (20) have removable singularities at integer multiples of π which we denote by $n\pi$. These singularities appear in the expressions $I(\theta)$, $\cot\theta$, and in the integrals found in Eqs. (15). Computation is troublesome at or near these removable singularities.

If we integrate Eq. (1) by parts, it becomes

$$I(\theta) = 2eJ(\theta) - \frac{\cot\theta}{r(\theta)^2} + c \qquad (21)$$

which holds except at the singularities of $I(\theta)$, where c is a constant of integration and

$$J(\theta) = \int_{\theta_0}^{\theta} \frac{\cos\zeta}{r(\zeta)^3}\,\mathrm{d}\zeta \qquad (22)$$

It is observed that the integral $J(\theta)$ has no singularities. It follows from the continuity of $J(\theta)$ that

$$2eJ(n\pi) + c = \lim_{\theta\to n\pi}\left[I(\theta) + \frac{\cot\theta}{r(\theta)}\right] \qquad (23)$$

so that the singularities in Eqs. (11), (15), and (20) that involve $I(\theta)$ are removed. With these changes the primer vector defined by Eq. (11) becomes

$$Q_1(\theta) = -r(\theta)[c_1 + 2c_2eJ(\theta)] - c_3\frac{[1+r(\theta)]}{r(\theta)}\sin\theta + \frac{c_4}{r(\theta)} \qquad (24a)$$

$$Q_2(\theta) = c_1e\sin\theta + c_2e\left[2eJ(\theta)\sin\theta - \frac{\cos\theta}{r(\theta)^2}\right] - c_3\cos\theta \qquad (24b)$$

$$Q_3(\theta) = \frac{c_5\cos\theta + c_6\sin\theta}{r(\theta)} \qquad (24c)$$

where the constant of integration c is absorbed by the constant c_2. This expression for the primer vector has no singularities or computational difficulties. The two integrals that appear in Eqs. (15a) and (15b) are replaced as follows:

$$\int_{\theta_0}^{\theta}\frac{L(\zeta)}{r(\zeta)^2}\csc^2\zeta\,\mathrm{d}\zeta = -\frac{L(\theta)\cot\theta}{r(\theta)^2} + J_1(\theta) + c_7 \qquad (25)$$

$$\int_{\theta_0}^{\theta}\left[\frac{L(\zeta)}{e}\csc^2\zeta + r_1(\zeta)\right]\mathrm{d}\zeta = -\frac{L(\theta)}{e}\cot\theta + J_2(\theta) + c_8 \qquad (26)$$

where c_7 and c_8 are constants of integration and

$$J_1(\theta) = \int_{\theta_0}^{\theta}\left[\frac{a_2(\zeta) - 2r_1(\zeta)r(\zeta)^2 + 2eL(\zeta)}{r(\zeta)^3}\right]\cos\zeta\,\mathrm{d}\zeta \qquad (27)$$

$$J_2(\theta) = \int_{\theta_0}^{\theta}\left[\frac{1}{e}[a_2(\zeta) - 2r_1(\zeta)]r(\zeta)\cos\zeta + r_1(\zeta)\right]\mathrm{d}\zeta \qquad (28)$$

Although similar notation is used, $J_1(\theta)$ and $J_2(\theta)$ are not related to $J(\theta)$, except that these two integrals are also free of singularities. With this replacement, some rearrangement of terms, and absorption of the constants c_7 and c_8, Eqs. (15) become

$$y_1(\theta) = -r(\theta)^2[b_1 + 2b_2eJ(\theta) + J_1(\theta)/e] + J_2(\theta) + b_4 \qquad (29a)$$

$$y_2(\theta) = r(\theta)\sin\theta[b_1e + 2b_2e^2J(\theta) + J_1(\theta)] - [\cos\theta/r(\theta)]\,[b_2e + L(\theta)] \qquad (29b)$$

$$y_3(\theta) = [b_5 + L_1(\theta)]\cos\theta + [b_6 + L_2(\theta)]\sin\theta \qquad (29c)$$

These equations are preferred over Eqs. (15). The integrals $J_1(\theta)$, $J_2(\theta)$, $L(\theta)$, $L_1(\theta)$, and $L_2(\theta)$ cannot be evaluated in closed form for powered flight.

For unpowered flight, $\boldsymbol{a}(\theta)$ is identically zero in Eqs. (16–19) simplifying Eqs. (27) and (28), and Eqs. (29) become

$$y_1(\theta) = -r(\theta)^2[b_1 + 2b_2eJ(\theta)] - b_3[1 + r(\theta)]\sin\theta + b_4 \qquad (30a)$$

$$y_2(\theta) = r(\theta)\sin\theta[b_1e + 2b_2e^2J(\theta)] - \cos\theta\left[\frac{b_2e}{r(\theta)} + b_3r(\theta)\right] \qquad (30b)$$

$$y_3(\theta) = b_5\cos\theta + b_6\sin\theta \qquad (30c)$$

These equations are valid for all noncircular Keplerian orbits. The integral $J(\theta)$ can be evaluated in closed form, and the particular form is determined from the type of orbit. Equations (30) are more useful than Eq. (20).

Evaluation of $J(\theta)$

We show here that the integral $J(\theta)$ can be evaluated easily if we transform from the true anomaly θ to the eccentric anomaly E for elliptical orbits or its analog H for hyperbolic orbits.

Elliptical Orbits

For orbits in which $0 < e < 1$, we have the relationship between the eccentric anomaly E and true anomaly θ given by

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E} \qquad (31)$$

where $\sin\theta$ and $\sin E$ always have the same algebraic sign. Changing the variable to E in Eq. (22) establishes the much simpler integral

$$J(\theta) = (1-e^2)^{-5/2}\int_{E_0}^{E}(1 - e\cos\xi)(\cos\xi - e)\,\mathrm{d}\xi \qquad (32)$$

where E_0 is the eccentric anomaly at θ_0. This integral is easily evaluated using elementary methods to obtain

$$J(\theta) = -(1-e^2)^{-5/2}\left[\frac{3e}{2}E - (1+e^2)\sin E + \frac{e}{2}\sin E\cos E + C\right] \qquad (33)$$

where C is an arbitrary constant.

Hyperbolic Orbits

In a similar way we can evaluate $J(\theta)$ for orbits in which $e > 1$. We introduce the analog of the eccentric anomaly H by the relationship

$$\cos\theta = \frac{e - \cosh H}{e\cosh H - 1} \qquad (34)$$

where $\sin\theta$ and $\sinh H$ always have the same algebraic sign. With this substitution also the integral defined by Eq. (22) is easily evaluated. The result is

$$J(\theta) = -(e^2-1)^{-5/2}\times\left[\frac{3e\,H}{2} - (1+e^2)\sinh H + \frac{e\sinh H\cosh H}{2} + C\right] \qquad (35)$$

where again C denotes an arbitrary constant.

Parabolic Orbits

For the case in which $e = 1$, the integral $J(\theta)$ can be evaluated directly using the identity $\cos\theta = 2\cos^2(\theta/2) - 1$. The result is

$$J(\theta) = (1/4)\ \tan(\theta/2) - (1/20)\tan^5(\theta/2) + C \tag{36}$$

where C is again an arbitrary constant. Since this expression is only defined on the region $r(\theta) > 0$ (i.e., $\cos\theta > -1$), it has no singularities.

Conclusion

The optimal rendezvous equations for a spacecraft near a noncircular Keplerian orbit can be put in a form that avoids the computationally unstable removable singularities found in some earlier papers and the restriction to elliptical orbits found in others. These equations are identical for all noncircular orbits except in the evaluation of an integral whose form is determined by the type of orbit.

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Optimal Aeroassisted Orbital Plane Change with Heating-Rate Constraint

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Introduction

SINCE the pioneering work by London[1] on the use of aerodynamic forces to assist in the orbital plane change, some classical analyses have done by Dickmanns,[2] Vinh,[3] Vinh and Hanson,[4] Walberg,[5] and Miele and Vekataraman.[6] Furthermore, in the recent paper by Miele et al.,[7] several problems have been solved. The sequential gradient-restoration algorithm is used and the purpose of each problem is to minimize, for example, the energy required for orbital transfer, peak dynamic pressure, or peak heating rate, with a prescribed atmospheric plane change. The purpose of this Note is to investigate the optimal trajectories for aeroassisted orbital plane change subject to a heating-rate constraint when the plane-change angle is being maximized.

In this Note, we use the modified Chapman variables[3] to derive the set of dimensionless equations of motion and the dimensionless heating rate. Hence, it suffices to just specify the maximum lift-to-drag ratio $(L/D)_{max}$ as the sole physical characteristic of the vehicle. As a result of the variational formulation, we have a two-point boundary-value problem (TPBVP) in which the constraint forms an interior boundary condition. In order to circumvent the difficulties in numerical computation, the multiple shooting method and the continuation method[8] are used. The modified Newton method is used to induce and accelerate the convergence.

Dimensionless Equations of Motion and Heating Rate

The motion of the re-entry vehicle, considered as a mass point, is defined by the six variables r (range), θ (longitude), ϕ (latitude), V (velocity), γ (flight-path angle), and ψ (heading angle) as shown in Ref. 3.

Using a parabolic drag polar of the form

$$C_D = C_{DO} + KC_L^2 \tag{1}$$

we define the normalized lift coefficient

$$\lambda = C_L/C_L^* \tag{2}$$

where C_L^* is the lift coefficient corresponding to the maximum lift-to-drag ratio E^*. With given values of C_{DO} and K assumed constant at hypersonic speed, we can easily compute the values of C_L^*, C_D^*, and E^*. The Earth is assumed at rest and with a locally exponential atmosphere,

$$\mathrm{d}\rho/\rho = -\beta \mathrm{d}r \tag{3}$$

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